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Representations of Meaning

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Logic in Ancient Greece ($\sim$ 300 BC)

Aristotle's famous example:

- All men are mortal
- Socrates is a man
- Therefore, Socrates is mortal

Computations Linguistics (1950s)

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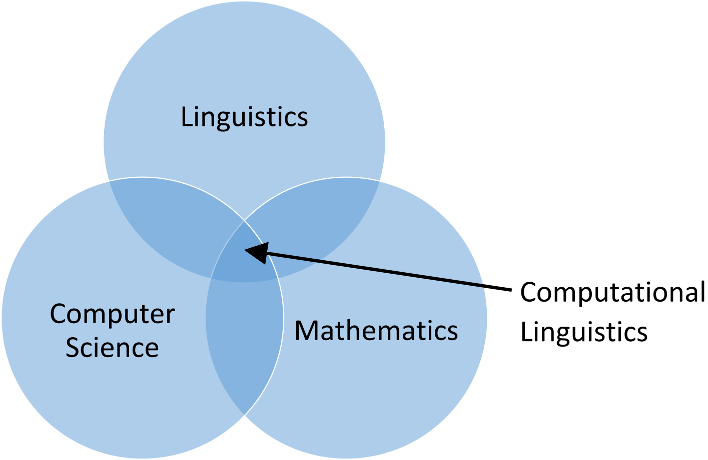

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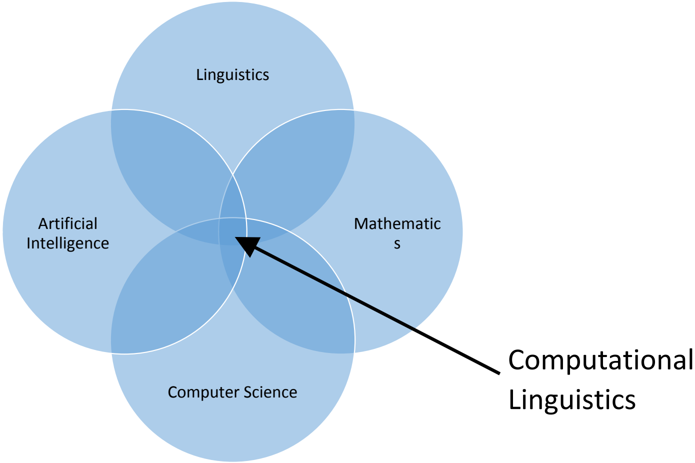

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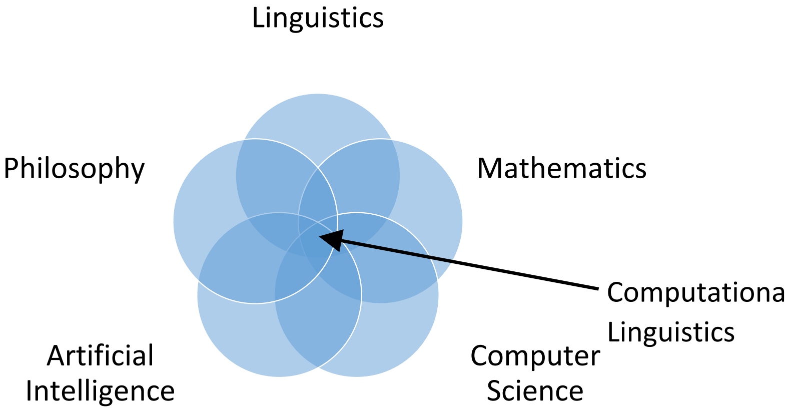

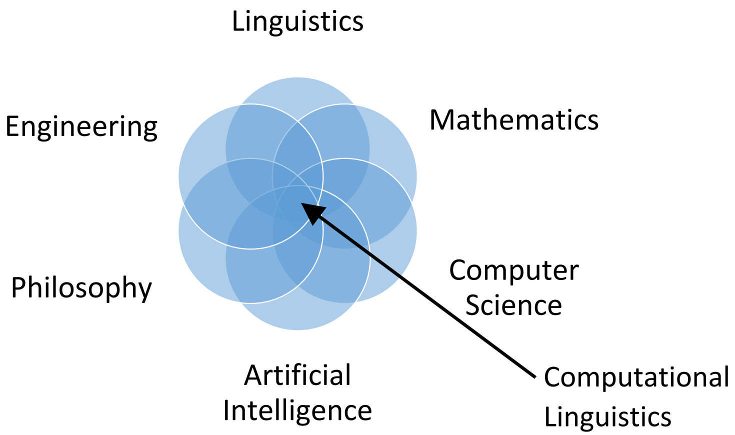
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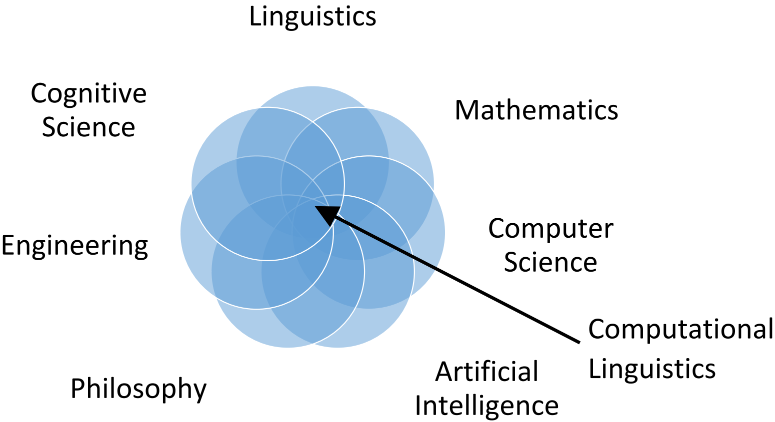

Gentzen Sequent Calculus (1934)

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- $A_1, ..., A_n \vdash A$: A can be proved from assumptions $A_1, ..., A_n$
- Γ, Δ: finite lists of formulas, define $\Gamma, A := \Gamma \cup \{A\}$
- Identity and cut rule—

$$\frac{}{A \vdash A}\text{Id} \qquad \frac{\Gamma \vdash A; \quad A, \Delta \vdash B}{\Gamma, \Delta \vdash B}\text{Cut}$$

- Conjunction—

$$\frac{\Gamma \vdash A; \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \wedge B} \wedge R \qquad \frac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C} \wedge L$$

- Implication—

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B} \to R \qquad \frac{\Gamma \vdash A; \quad B, \Delta \vdash C}{\Gamma, A \to B, \Delta \vdash C} \to L$$

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- **Equivalent to the Natural Deduction system.**

(based on notes by Samson Abramsky)

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Gentzen Sequent Calculus – Example

- The sentences “All men are mortal” and “Socrates is a man” are mapped to

$$\text{man} \vdash \text{mortal}; \quad \text{Socrates} \vdash \text{man}.$$

  Using the cut rule we get

$$\frac{\text{man} \vdash \text{mortal}; \quad \text{Socrates} \vdash \text{man}}{\text{Socrates} \vdash \text{mortal}}\text{Cut}$$

  And infer that Socrates is mortal.

- But, what about

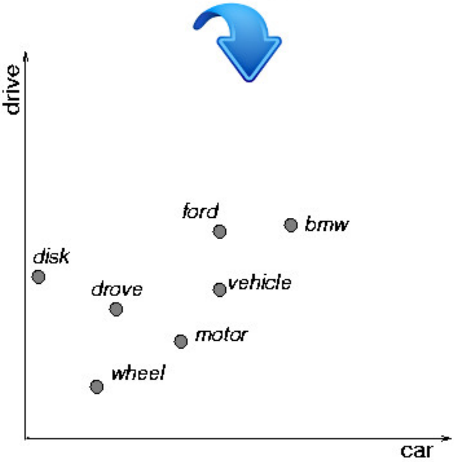
$$\text{went} \rightarrow \text{pub} \overset{?}{\vdash} \text{went} \rightarrow \text{bar}$$

You shall know a word by the company it keeps.

–Firth, J. R. 1957:11

Vector space over $\mathbb{R}$

$$
\begin{array}{l}
\\
\textit{term } 1 \\
\textit{term } 2 \\
\\
\textit{term } 300{,}000
\end{array}
\begin{array}{c}
\begin{array}{cccccccc}
d_1 & d_2 & d_3 & d_4 & d_5 & d_6 & & d_{3,000,000,000}
\end{array} \\
\begin{pmatrix}
0 & 0 & 1 & 0 & 1 & 0 & \cdots & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & 1 & 0 & \cdots & 0
\end{pmatrix}
\end{array}
$$

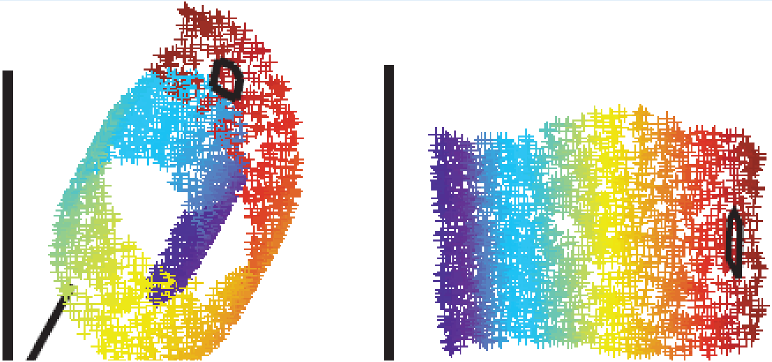


The Johnson-Lindenstrauss Lemma

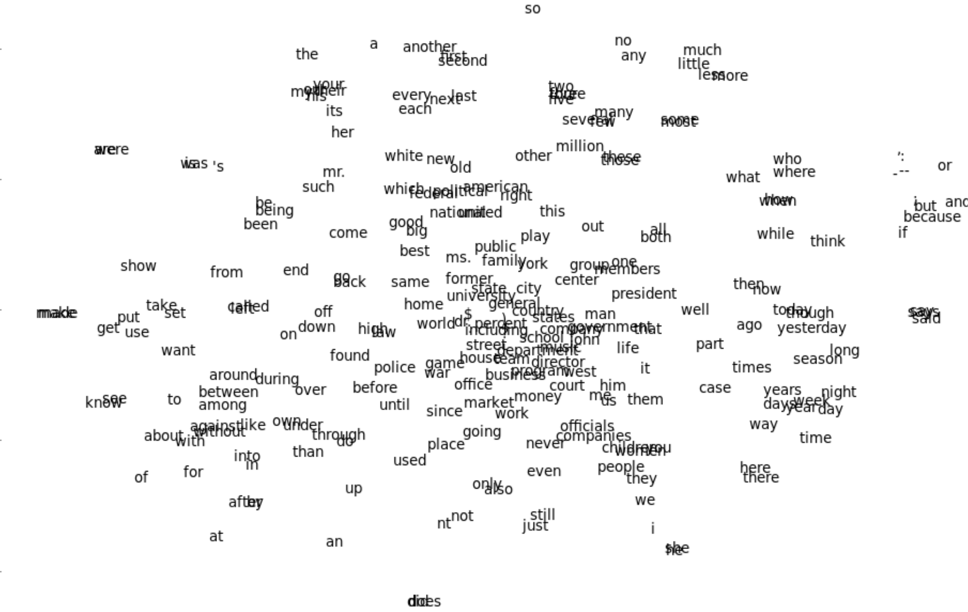
For any $0 < \epsilon < 1/2$ and any integer $m > 4$, let $k = \frac{20 \log m}{\epsilon^2}$. Then, for any set V of m points in $\mathbb{R}^N$, $\exists f : \mathbb{R}^N \to \mathbb{R}^k$ s.t. $\forall u, v \in V$:

$$(1\epsilon)||uv||^2 \leq ||f(u)f(v)||^2 \leq (1+\epsilon)||uv||^2$$

e.g. $f(x) = \frac{1}{\sqrt{k}} Ax$ with $A_{ij} \in \mathcal{N}(0,1)$.



Vector space over $\mathbb{R}$ – Example



A category **C** consists of:

- A set *ob*(**C**) of objects,
- A set *hom*(**C**) of morphisms, or arrows. Each arrow f has a source object A and target object B,
- An identity arrow id_A for every object A.
- An associative composition operation between arrows $\circ$.

Composing $f : A \rightarrow B$ and $g : B \rightarrow C$ gives $g \circ f$ from A to C.

A **symmetric monoidal closed category** is

- a category **C**
- equipped with a symmetric associative bifunctor $\otimes : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ called the tensor product
- and an object I called the identity object[1].
- For all objects A and B there is an object

  $$A \multimap B$$

  and an arrow

  $$ev_{A,B} : (A \multimap B) \otimes A \to B.$$

  For every arrow $f : C \otimes A \to B$, there is a unique arrow $\Lambda(f) : C \to (A \multimap B)$ such that $ev_{A,B} \circ (\Lambda(f) \otimes \mathrm{id}_A) = f$.

[1] The symmetry, associativity and identity are define through natural isomorphisms.

Symmetric monoidal closed categories

The category of **finite dimensional vector spaces** is a symmetric monoidal closed category.

- The tensor product is the tensor product of vector spaces,
- and $A \multimap B$ is the vector space of linear maps.

Gentzen sequent calculus without the Contraction and Weakening rules also corresponds to symmetric monoidal closed categories.

- Known as **linear logic**,
- A “resource-sensitive” logic[2],
- The tensor product is the conjunction $\wedge$,
- and $A \multimap B$ is the implication $\rightarrow$.

For example, the identity and cut rule —

$$\frac{}{A \vdash A}\text{Id} \qquad \frac{\Gamma \vdash A; \quad A, \Delta \vdash B}{\Gamma, \Delta \vdash B}\text{Cut}$$

are equivalent to

$$\frac{}{\text{Id}_A : A \rightarrow A} \qquad \frac{f : \Gamma \rightarrow A; \quad g : A \otimes \Delta \rightarrow B}{g \circ (f \otimes \text{Id}_\Delta) : \Gamma \otimes \Delta \rightarrow B}$$

[2] It is possible to recover the expressive power of standard Gentzen sequent calculus with the addition of some connectives.

The Compositional Distributional representation:

- Let $N = \mathbb{R}$ be a *noun vector space*, with nouns represented as vectors,
- Let $V = \mathbb{R}^3$ be a *verb vector space*, with verbs represented as third-order tensors,
- Let S be a *sentence vector space*, representing sentences as the tensor product $N \otimes V \otimes N$,
- Finally let T be a *truth value vector space*, a lower dimensional vector space to which we project products.

For example,

- "Dogs" and "Cats" are represented as vectors d and c
- "chase" is represented as $T \in \mathbb{R}^3$
- "Dogs chase cats" is represented as $d \otimes T \otimes c$

For simplicity, we use a **binary noun vector space**, and T is a binary matrix:

- The first dimension of N is "*likes chasing small fluffy animals*" and the 2nd and 3rd dimensions are "*is small*" and "*is fluffy*",
- A cat is represented as small and fluffy $c = [0, 1, 1]$, a dog likes to chase small and fluffy animals $d = [1, 0, 0]$,
- Tensor "chase" preserves vectors from the left that have the "likes chasing small fluffy animals" property and vectors from the right that have the "is small and is fluffy properties,

$$T = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Compositional Distributional – Example

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$$c = [0, 1, 1], \quad d = [1, 0, 0]$$

$$T = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad t = dTc^T$$

- The proposition “dogs chase cats” is mapped to $dTc^T = 1$ – a high “truthness” value.
- The proposition “cats chase dogs” is mapped to $cTd^T = 0$ – low truthness.

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The talk was based on [Gal, 2013].

Gal, Y. (2013).
Semantics, modelling, and the problem of representation of meaning – a brief survey of recent literature.
Technical report, University of Cambridge.